

Crossing of the phantom divide in modified gravity

Kazuharu Bamba^{1,*}, Chao-Qiang Geng^{1,†}, Shin'ichi Nojiri^{2,‡} and Sergei D. Odintsov^{3,§}

¹*Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300*

²*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

³*Institució Catalana de Recerca i Estudis Avançats*

(ICREA) and Institut de Ciències de l'Espai (IEEC-CSIC),

Campus UAB, Facultat de Ciències, Torre C5-Par-2a pl,

E-08193 Bellaterra (Barcelona), Spain

Abstract

We reconstruct an explicit model of modified gravity in which a crossing of the phantom divide can be realized. It is shown that the (finite-time) Big Rip singularity appears in the model of modified gravity (i.e., in the so-called Jordan frame), whereas that in the corresponding scalar field theory obtained through the conformal transformation (i.e., in the so-called Einstein frame) the singularity becomes the infinite-time one. Furthermore, we investigate the relations between the scalar field theories with realizing a crossing of the phantom divide and the corresponding modified gravitational theories by using the inverse conformal transformation. It is demonstrated that the scalar field theories describing the non-phantom phase (phantom one with the Big Rip) can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation. We also study a viable model of modified gravity in which the transition from the de Sitter universe to the phantom phase can occur. In addition, we explore the stability for the obtained solutions of the crossing of the phantom divide under a quantum correction coming from conformal anomaly.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k

* E-mail address: bamba“at”phys.nthu.edu.tw

† E-mail address: geng“at”phys.nthu.edu.tw

‡ E-mail address: nojiri“at”phys.nagoya-u.ac.jp

§ Also at Lab. Fundam. Study, Tomsk State Pedagogical University, Tomsk. E-mail address: odintsov“at”aliga.ieec.uab.es

I. INTRODUCTION

It is observationally confirmed that the current expansion of the universe is accelerating [1, 2, 3]. Various scenarios to explain the current accelerated expansion of the universe have been proposed. The mechanism, however, is not well understood yet (for recent reviews, see [4, 5, 6, 7, 8]).

There are two approaches to account for the current accelerated expansion of the universe. One is to introduce some unknown matter, which is called “dark energy” in the framework of general relativity. The other is to modify the gravitational theory, e.g., in simplest case to study the action described by an arbitrary function of the scalar curvature R , which is called “ $F(R)$ gravity”. Here, $F(R)$ is an arbitrary function of the scalar curvature R (for reviews, see [7, 8]).

According to the recent various observational data including the Type Ia supernovae Gold dataset [9], there exists the possibility that the effective equation of state (EoS) parameter, which is the ratio of the effective pressure of the universe to the effective energy density of it, evolves from larger than -1 (non-phantom phase) to less than -1 (phantom one, in which superacceleration is realized; e.g., see [10]), namely, crosses -1 (the phantom divide) currently or in near future.

A number of attempts to realize the crossing of the phantom divide have been made in the framework of general relativity: For instance, scalar-tensor theories with the non-minimal gravitational coupling between a scalar field and the scalar curvature [11] or that between a scalar field and the Gauss-Bonnet term [12], one scalar field model with non-linear kinetic terms [13] or a non-linear higher-derivative one [14], phantom coupled to dark matter with an appropriate coupling [15], the thermodynamical inhomogeneous dark energy model [16], multiple kinetic k-essence [17], multi-field models (two scalar fields model [18, 19, 20], “quintom” consisting of phantom and canonical scalar fields [21]), and the description of those models through the Parameterized Post-Friedmann approach [22], or a classical Dirac field [23] or string-inspired models [24], non-local gravity [25, 26], a model in loop quantum cosmology [27] and a general consideration of the crossing of the phantom divide [28, 29, 30] (for a detailed review, see [6]). In fact, however, explicit models of modified gravity realizing the crossing of the phantom divide have hardly been investigated, although there were suggestive and interesting related works [7, 31, 32].

In the present paper, we study a crossing of the phantom divide in modified gravity. We reconstruct an explicit model of modified gravity in which a crossing of the phantom divide can be realized by using the reconstruction method proposed in Ref. [33]. Furthermore, we investigate the corresponding scalar field theory, in which there exist the Einstein-Hilbert action and a scalar field, obtained through a conformal transformation of the modified gravitational theory, and compare the evolution of the universe in the modified gravitational theory with that in the corresponding scalar field theory. It is shown that the (finite-time) Big Rip singularity [34, 35] appears in the reconstructed model of modified gravity (i.e., in the so-called Jordan frame), whereas that in the corresponding scalar field theory obtained through the conformal transformation (i.e., in the so-called Einstein frame) the singularity becomes the infinite-time one. Moreover, we consider the relations between the scalar field theories with realizing a crossing of the phantom divide and the corresponding theories of modified gravity by using the inverse conformal transformation of scalar field theories. It is demonstrated that the scalar field theories describing the non-phantom phase (phantom one with the Big Rip singularity) can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation. On the other hand, a very realistic model of modified gravity that evades solar-system tests and realizes a viable cosmic expansion in the past has recently been proposed in Ref. [36] (for some related models, see [37, 38]). In this model, our universe is asymptotically de Sitter space. Therefore, we also reconstruct a model of modified gravity in which the transition from the de Sitter universe to the phantom phase can occur in such a viable theory. In addition, we explore the stability for the obtained solutions of the crossing of the phantom divide under a quantum correction, in particular conformal anomaly.

Our goal in this paper is to show that in principle the crossing of the phantom divide can be realized in the framework of modified gravity without introducing any extra scalar components with the wrong kinetic sign (phantom). We reconstruct such an explicit model of modified gravity. By presenting it, it can be illustrated that the crossing of the phantom divide can occur in modified gravity as the scalar field theories in the framework of general relativity. The demonstration in this work can be interpreted as a meaningful step to construct a more realistic model of modified gravity, which could correctly describe the expansion history of the universe.

This paper is organized as follows. In Sec. II we explain the reconstruction method of

modified gravity proposed in Ref. [33]. Using this method, we reconstruct an explicit model of modified gravity in which a crossing of the phantom divide can be realized. In particular, we show that the Big Rip singularity appears in this modified gravitational theory. In Sec. III we consider the corresponding scalar field theory, which is obtained by making the conformal transformation of the modified gravitational theory with realizing a crossing of the phantom divide. We demonstrate that the Big Rip singularity does not appear in the corresponding scalar field theory. In Sec. IV we investigate the relations between scalar field theories and the corresponding modified gravitational ones. In Sec. V we study the viable model of modified gravity in which the transition from the de Sitter universe to the phantom phase can occur. In Sec. VI, we examine the stability for the obtained solutions of the phantom crossing under a quantum correction coming from conformal anomaly. Finally, some summaries and outlooks are given in Sec. VII. Detailed derivations and explanations about each section are shown in Appendixes A–F. We use units in which $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by κ^2 , so that $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$, where $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \text{GeV}$ is the Planck mass.

II. RECONSTRUCTION OF MODIFIED GRAVITY

We investigate modified gravity with realizing a crossing of the phantom divide by using the reconstruction method. (The equivalence between $F(R)$ gravity and the scalar tensor theory was explicitly shown in Ref. [39]. The limited case was given in Ref. [40].)

A. Reconstruction method

First, we briefly review the reconstruction method of modified gravity proposed in Ref. [33].

The action of $F(R)$ gravity with general matter is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right], \quad (2.1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian.

The action (2.1) can be rewritten to the following form by using proper functions $P(\phi)$

and $Q(\phi)$ of a scalar field ϕ :

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} [P(\phi)R + Q(\phi)] + \mathcal{L}_{\text{matter}} \right\}. \quad (2.2)$$

The scalar field ϕ may be regarded as an auxiliary scalar field because ϕ has no kinetic term. It follows from the action (2.1) that the equation of motion of ϕ is given by

$$0 = \frac{dP(\phi)}{d\phi} R + \frac{dQ(\phi)}{d\phi}, \quad (2.3)$$

which may be solved with respect to ϕ as $\phi = \phi(R)$. Substituting $\phi = \phi(R)$ into the action (2.2), we find that the expression of $F(R)$ in the action of $F(R)$ gravity in Eq. (2.1) is given by

$$F(R) = P(\phi(R))R + Q(\phi(R)). \quad (2.4)$$

From the action (2.2), we find that the field equation of modified gravity is given by

$$\frac{1}{2}g_{\mu\nu} [P(\phi)R + Q(\phi)] - R_{\mu\nu}P(\phi) - g_{\mu\nu}\square P(\phi) + \nabla_\mu \nabla_\nu P(\phi) + \kappa^2 T_{\mu\nu}^{(\text{matter})} = 0, \quad (2.5)$$

where ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\square \equiv g^{\mu\nu}\nabla_\mu \nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $T_{\mu\nu}^{(\text{matter})}$ is the contribution to the matter energy-momentum tensor.

We assume the flat Friedmann-Robertson-Walker (FRW) space-time with the metric,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad (2.6)$$

where $a(t)$ is the scale factor.

In this background, the $(\mu, \nu) = (0, 0)$ component and the trace part of the $(\mu, \nu) = (i, j)$ component of Eq. (2.5), where i and j run from 1 to 3, read

$$-6H^2 P(\phi(t)) - Q(\phi(t)) - 6H \frac{dP(\phi(t))}{dt} + 2\kappa^2 \rho = 0, \quad (2.7)$$

and

$$2 \frac{d^2 P(\phi(t))}{dt^2} + 4H \frac{dP(\phi(t))}{dt} + (4\dot{H} + 6H^2) P(\phi(t)) + Q(\phi(t)) + 2\kappa^2 p = 0, \quad (2.8)$$

respectively, where $H = \dot{a}/a$ is the Hubble parameter and a dot denotes a time derivative, $\cdot = \partial/\partial t$. Here, ρ and p are the sum of the energy density and pressure of matters with a constant EoS parameter w_i , respectively, where i denotes some component of the matters.

Eliminating $Q(\phi)$ from Eqs. (2.7) and (2.8), we obtain

$$\frac{d^2 P(\phi(t))}{dt^2} - H \frac{dP(\phi(t))}{dt} + 2\dot{H}P(\phi(t)) + \kappa^2(\rho + p) = 0. \quad (2.9)$$

We note that the scalar field ϕ may be taken as $\phi = t$ because ϕ can be redefined properly.

We now consider that $a(t)$ is described as

$$a(t) = \bar{a} \exp(\tilde{g}(t)), \quad (2.10)$$

where \bar{a} is a constant and $\tilde{g}(t)$ is a proper function. In this case, Eq. (2.9) is reduced to

$$\begin{aligned} \frac{d^2 P(\phi)}{d\phi^2} - \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} + 2 \frac{d^2 \tilde{g}(\phi)}{d\phi^2} P(\phi) \\ + \kappa^2 \sum_i (1 + w_i) \bar{\rho}_i \bar{a}^{-3(1+w_i)} \exp[-3(1 + w_i) \tilde{g}(\phi)] = 0, \end{aligned} \quad (2.11)$$

where $\bar{\rho}_i$ is a constant and we have used $H = d\tilde{g}(\phi)/(d\phi)$. Moreover, it follows from Eq. (2.7) that $Q(\phi)$ is given by

$$\begin{aligned} Q(\phi) = -6 \left[\frac{d\tilde{g}(\phi)}{d\phi} \right]^2 P(\phi) - 6 \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} \\ + 2\kappa^2 \sum_i \bar{\rho}_i \bar{a}^{-3(1+w_i)} \exp[-3(1 + w_i) \tilde{g}(\phi)]. \end{aligned} \quad (2.12)$$

Hence, if we obtain the solution of Eq. (2.11) with respect to $P(\phi)$, then we can find $Q(\phi)$. In Appendix A, some points on the reconstruction method are noted.

We mention that the convenient reconstruction for scalar field theories could be given in Refs. [41, 42] (for a recent review, see [43]). Furthermore, the reconstruction in the scalar-Einstein-Gauss-Bonnet theories was considered in Ref. [44].

B. Explicit model with realizing a crossing of the phantom divide

Next, using the reconstruction method explained in the preceding subsection, we reconstruct an explicit model in which a crossing of the phantom divide can be realized.

A solution of Eq. (2.11) without matter can be given by

$$P(\phi) = e^{\tilde{g}(\phi)/2} \tilde{p}(\phi), \quad (2.13)$$

$$\tilde{g}(\phi) = -10 \ln \left[\left(\frac{\phi}{t_0} \right)^{-\gamma} - C \left(\frac{\phi}{t_0} \right)^{\gamma+1} \right], \quad (2.14)$$

$$\tilde{p}(\phi) = \tilde{p}_+ \phi^{\beta_+} + \tilde{p}_- \phi^{\beta_-}, \quad (2.15)$$

$$\beta_{\pm} = \frac{1 \pm \sqrt{1 + 100\gamma(\gamma + 1)}}{2}, \quad (2.16)$$

where γ and C are positive constants, t_0 is the present time, and \tilde{p}_\pm are arbitrary constants. The derivation of this solution is shown in Appendix B.

From Eq. (2.14), we find that $\tilde{g}(\phi)$ diverges at finite ϕ when

$$\phi = t_s \equiv t_0 C^{-1/(2\gamma+1)}, \quad (2.17)$$

which tells that there could be the Big Rip singularity at $t = t_s$ [34, 35]. (Other kinds of finite-time future singularities have been studied in Ref. [45].) One only needs to consider the period $0 < t < t_s$ because $\tilde{g}(\phi)$ should be real number. Eq. (2.14) also gives the following Hubble rate $H(t)$:

$$H(t) = \frac{d\tilde{g}(\phi)}{d\phi} = \left(\frac{10}{t_0}\right) \left[\frac{\gamma \left(\frac{\phi}{t_0}\right)^{-\gamma-1} + (\gamma+1)C \left(\frac{\phi}{t_0}\right)^\gamma}{\left(\frac{\phi}{t_0}\right)^{-\gamma} - C \left(\frac{\phi}{t_0}\right)^{\gamma+1}} \right], \quad (2.18)$$

where it is taken $\phi = t$.

In the FRW background (2.6), even for modified gravity described by the action (2.1), the effective energy-density and pressure of the universe are given by $\rho_{\text{eff}} = 3H^2/\kappa^2$ and $p_{\text{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2$, respectively. The effective EoS parameter $w_{\text{eff}} = p_{\text{eff}}/\rho_{\text{eff}}$ is defined as [7]

$$w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2}. \quad (2.19)$$

For the case of $H(t)$ in Eq. (2.18), from Eq. (2.19) we find that w_{eff} is expressed as

$$w_{\text{eff}} = -1 + U(t), \quad (2.20)$$

where

$$U(t) \equiv -\frac{2\dot{H}}{3H^2} = -\frac{-\gamma + 4\gamma(\gamma+1) \left(\frac{t}{t_s}\right)^{2\gamma+1} + (\gamma+1) \left(\frac{t}{t_s}\right)^{2(2\gamma+1)}}{15 \left[\gamma + (\gamma+1) \left(\frac{t}{t_s}\right)^{2\gamma+1} \right]^2}. \quad (2.21)$$

Moreover, the scalar curvature is given by $R = 6(\dot{H} + 2H^2)$. For the case of Eq. (2.18), R is described as

$$R = \frac{60 \left[\gamma(20\gamma-1) + 44\gamma(\gamma+1) \left(\frac{t}{t_s}\right)^{2\gamma+1} + (\gamma+1)(20\gamma+21) \left(\frac{t}{t_s}\right)^{2(2\gamma+1)} \right]}{t^2 \left[1 - \left(\frac{t}{t_s}\right)^{2\gamma+1} \right]^2}. \quad (2.22)$$

In deriving Eqs. (2.21) and (2.22), we have used Eq. (2.17).

When $t \rightarrow 0$, i.e., $t \ll t_s$, $H(t)$ behaves as

$$H(t) \sim \frac{10\gamma}{t} . \quad (2.23)$$

In this limit, it follows from Eq. (2.19) that the effective EoS parameter is given by

$$w_{\text{eff}} = -1 + \frac{1}{15\gamma} . \quad (2.24)$$

This behavior is identical with that in the Einstein gravity with matter whose EoS parameter is greater than -1 .

On the other hand, when $t \rightarrow t_s$, we find

$$H(t) \sim \frac{10}{t_s - t} . \quad (2.25)$$

In this case, the scale factor is given by $a(t) \sim \bar{a} (t_s - t)^{-10}$. When $t \rightarrow t_s$, therefore, $a \rightarrow \infty$, namely, the Big Rip singularity appears. In this limit, the effective EoS parameter is given by

$$w_{\text{eff}} = -1 - \frac{1}{15} = -\frac{16}{15} . \quad (2.26)$$

This behavior is identical with the case in which there is a phantom matter with its EoS parameter being smaller than -1 . Thus, we have obtained an explicit model showing a crossing of the phantom divide.

It follows from Eq. (2.19) that the effective EoS parameter w_{eff} becomes -1 when $\dot{H} = 0$. Solving $w_{\text{eff}} = -1$ with respect to t by using Eq. (2.20), namely, $U(t) = 0$, we find that the effective EoS parameter crosses the phantom divide at $t = t_c$ given by

$$t_c = t_s \left(-2\gamma + \sqrt{4\gamma^2 + \frac{\gamma}{\gamma+1}} \right)^{1/(2\gamma+1)} . \quad (2.27)$$

From Eq. (2.21), we see that when $t < t_c$, $U(t) > 0$ because $\gamma > 0$. Moreover, the time derivative of $U(t)$ is given by

$$\frac{dU(t)}{dt} = -\frac{2\gamma(\gamma+1)(2\gamma+1)^2}{15 \left[\gamma + (\gamma+1) \left(\frac{t}{t_s} \right)^{2\gamma+1} \right]^3} \left(\frac{1}{t_s} \right) \left(\frac{t}{t_s} \right)^{2\gamma} \left[1 - \left(\frac{t}{t_s} \right)^{2\gamma+1} \right] . \quad (2.28)$$

Eq. (2.28) tells that the relation $dU(t)/(dt) < 0$ is always satisfied because we only consider the period $0 < t < t_s$ as mentioned above. This means that $U(t)$ decreases monotonously.

Thus, the value of $U(t)$ evolves from positive to negative. From Eq. (2.20), we see that the value of w_{eff} crosses -1 . Once the universe enters the phantom phase, it stays in this phase, namely, the value of w_{eff} remains less than -1 , and finally the Big Rip singularity appears because $U(t)$ decreases monotonically. Note that other types of the finite-time future singularities in modified gravity are possible as demonstrated in Ref. [46].

It follows from Eqs. (2.13), (2.14), (2.15) and (2.17) that $P(t)$ is given by

$$P(t) = \left[\frac{\left(\frac{t}{t_0}\right)^\gamma}{1 - \left(\frac{t}{t_s}\right)^{2\gamma+1}} \right]^5 \sum_{j=\pm} \tilde{p}_j t^{\beta_j}. \quad (2.29)$$

Using Eqs. (2.12) and (2.29), one gets

$$Q(t) = -6H \left[\frac{\left(\frac{t}{t_0}\right)^\gamma}{1 - \left(\frac{t}{t_s}\right)^{2\gamma+1}} \right]^5 \sum_{j=\pm} \left(\frac{3}{2}H + \frac{\beta_j}{t} \right) \tilde{p}_j t^{\beta_j}. \quad (2.30)$$

If we can solve Eq. (2.22) with respect to t as $t = t(R)$, in principle we can obtain the form of $F(R)$ by using this solution and Eqs. (2.4), (2.29) and (2.30). In fact, however, for the general case it is difficult to solve Eq. (2.22) as $t = t(R)$. Hence, as an solvable example, we show the behavior of $t_s^2 F(\tilde{R})$ as a function of $\tilde{R} \equiv t_s^2 R$ in Fig. 1 for $\gamma = 1/2$, $\tilde{p}_+ = -1/t_s^{\beta_+}$, $\tilde{p}_- = 0$, $\beta_+ = (1 + 2\sqrt{19})/2$ and $t_s = 2t_0$. The quantities in Fig. 1 are shown in dimensionless quantities. The horizontal and vertical axes show \tilde{R} and $t_s^2 F$, respectively. (Here, $\tilde{R} = t_s^2 R = 4R/R_0$, where R_0 is the current curvature. In deriving this relation, we have used $t_s = 2t_0$, $t_0 \approx H_0^{-1}$, where H_0 is the present Hubble parameter.) From Fig. 1, we see that the value of $F(R)$ increases as that of R becomes larger.

To examine the analytic form of $F(R)$ for the general case, we investigate the behavior of $F(R)$ in the limits $t \rightarrow 0$ and $t \rightarrow t_s$. When $t \rightarrow 0$, from Eq. (2.23) we find

$$t \sim \sqrt{\frac{60\gamma(20\gamma-1)}{R}}. \quad (2.31)$$

In this limit, it follows from Eqs. (2.4), (2.23), (2.29), (2.30) and (2.31) that the form of $F(R)$ is given by

$$F(R) \sim \left\{ \frac{\left[\frac{1}{t_0} \sqrt{60\gamma(20\gamma-1)} R^{-1/2} \right]^\gamma}{1 - \left[\frac{1}{t_s} \sqrt{60\gamma(20\gamma-1)} R^{-1/2} \right]^{2\gamma+1}} \right\}^5 R \\ \times \sum_{j=\pm} \left\{ \left(\frac{5\gamma-1-\beta_j}{20\gamma-1} \right) \tilde{p}_j [60\gamma(20\gamma-1)]^{\beta_j/2} R^{-\beta_j/2} \right\}. \quad (2.32)$$

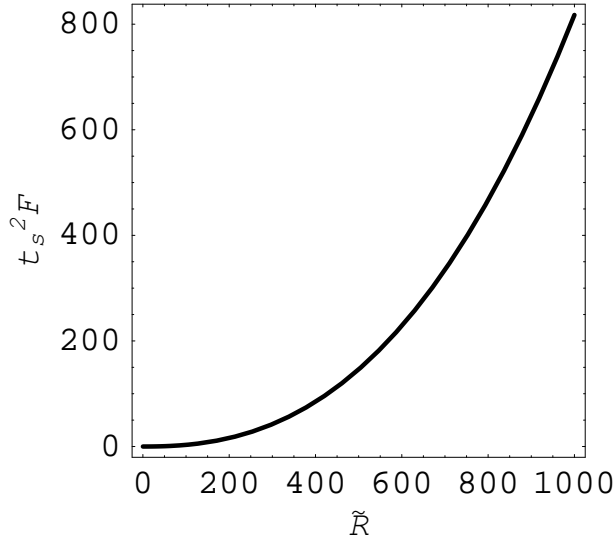


FIG. 1: Behavior of $t_s^2 F(\tilde{R})$ as a function of \tilde{R} for $\gamma = 1/2$, $\tilde{p}_+ = -1/t_s^{\beta_+}$, $\tilde{p}_- = 0$, $\beta_+ = (1 + 2\sqrt{19})/2$ and $t_s = 2t_0$.

On the other hand, when $t \rightarrow t_s$, from Eq. (2.25) we obtain

$$t \sim t_s - 3\sqrt{\frac{140}{R}}. \quad (2.33)$$

In this limit, it follows from Eqs. (2.4), (2.25), (2.29), (2.30) and (2.33) that the form of $F(R)$ is given by

$$F(R) \sim \left(\frac{\left\{ \frac{1}{t_0} [t_s - 3\sqrt{140}R^{-1/2}] \right\}^\gamma}{1 - \left[1 - \frac{3\sqrt{140}}{t_s} R^{-1/2} \right]^{2\gamma+1}} \right)^5 R \sum_{j=\pm} \tilde{p}_j [t_s - 3\sqrt{140}R^{-1/2}]^{\beta_j} \\ \times \left\{ 1 - \sqrt{\frac{20}{7}} \left[\sqrt{\frac{15}{84}} t_s + (\beta_j - 15) R^{-1/2} \right] \frac{1}{t_s - 3\sqrt{140}R^{-1/2}} \right\}. \quad (2.34)$$

The above modified gravity may be considered as some approximated form of more realistic viable theory. For large R , namely, $t_s^2 R \gg 1$, the expression of $F(R)$ in (2.34) can be approximately written as

$$F(R) \approx \frac{2}{7} \left[\frac{1}{3\sqrt{140}(2\gamma+1)} \left(\frac{t_s}{t_0} \right)^\gamma \right]^5 \left(\sum_{j=\pm} \tilde{p}_j t_s^{\beta_j} \right) t_s^5 R^{7/2}. \quad (2.35)$$

III. CORRESPONDING SCALAR FIELD THEORY

In this section, motivated by the discussion in Ref. [47], we consider the corresponding scalar field theory to modified gravity with realizing a crossing of the phantom divide, which is obtained by making the conformal transformation of the modified gravitational theory.

By introducing two scalar fields ζ and ξ , we can rewrite the action (2.1) to the following form [7]:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} [\xi (R - \zeta) + F(\zeta)] + \mathcal{L}_{\text{matter}} \right\}. \quad (3.1)$$

This is the action in the Jordan-frame, in which there exists a non-minimal coupling between ξ and the scalar curvature R . The form in Eq. (3.1) is reduced to the original one in Eq. (2.1) by using the equation $\zeta = R$, which is the equation of motion of one auxiliary field ξ .

We make the following conformal transformation:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = e^\sigma g_{\mu\nu}, \quad (3.2)$$

where

$$e^\sigma = F'(\zeta). \quad (3.3)$$

Here, σ is a scalar field and a hat denotes quantities in the Einstein frame, in which the non-minimal coupling between ξ and R in the action (3.1) disappears.

By defining φ as $\varphi \equiv \sqrt{3/2}\sigma/\kappa$, we obtain the following canonical scalar field theory:

$$S_{\text{ST}} = \int d^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) + e^{-2\sqrt{2/3}\kappa\varphi} \mathcal{L}_{\text{matter}} \right]. \quad (3.4)$$

The detailed derivation of the action (3.4) is given in Appendix C.

We now investigate the case in which $F(R)$ is given by

$$F(R) = c_1 M^2 \left(\frac{R}{M^2} \right)^{-n}, \quad (3.5)$$

where c_1 is a dimensionless constant and M denotes a mass scale. The form of $F(R)$ in Eq. (2.35) corresponds to the one in Eq. (3.5) with $n = -7/2$. It may seem that such a model may have problems in the description of the past universe evolution. However, there is the trick to make its past evolution consistent with observations described in Ref. [33]. It uses the introduction of compensating dark energy dominated at intermediate universe

which disappears effectively at current universe. Our primary purpose in this work is current universe admitting the phantom divide crossing in modified gravity, so we will not discuss the past evolution of the model under discussion. In this case, the scale factor $a(t)$ and the scalar curvature R are given by [47]

$$a(t) = \bar{a} (t_s - t)^{(n+1)(2n+1)/(n+2)}, \quad (3.6)$$

and

$$R = \frac{6n(n+1)(2n+1)(4n+5)}{(n+2)^2} \frac{1}{(t_s - t)^2}, \quad (3.7)$$

respectively. From Eqs. (3.2) and (3.3), we find

$$d\hat{t} = \pm e^{\sigma/2} dt, \quad (3.8)$$

$$e^{\sigma/2} = \sqrt{-nc_1} \left[\frac{(n+2)^2}{6n(n+1)(2n+1)(4n+5)} \right]^{(n+1)/2} M^{n+1} (t_s - t)^{n+1}, \quad (3.9)$$

where we have used Eq. (3.7). It follow from Eq. (3.9) that the relation between the cosmic time in the Einstein frame \hat{t} and that in the Jordan frame is given by

$$\hat{t} = \mp \frac{\sqrt{-nc_1}}{n+2} \left[\frac{(n+2)^2}{6n(n+1)(2n+1)(4n+5)} \right]^{(n+1)/2} M^{n+1} (t_s - t)^{n+2}. \quad (3.10)$$

If $n < -2$, the limit of $t \rightarrow t_s$ corresponds to that of $\hat{t} \rightarrow \mp\infty$. For the case of Eq. (2.35), $n = -7/2$. From Eqs. (3.2) and (3.3), we also find that the metric in the Einstein frame is expressed as

$$d\hat{s}^2 = e^{\sigma} ds^2 = -d\hat{t}^2 + \hat{a}(\hat{t}) d\mathbf{x}^2, \quad (3.11)$$

where $\hat{a}(\hat{t})$ is the scale factor in the scalar field theory given by

$$\hat{a}(\hat{t}) = \hat{\hat{a}} \hat{\hat{t}}^{3[(n+1)/(n+2)]^2}, \quad (3.12)$$

$$\begin{aligned} \hat{\hat{a}} &= \bar{a} \left(\mp \frac{1}{n+2} \right)^{-3[(n+1)/(n+2)]^2} \\ &\times \left\{ \sqrt{-nc_1} \left[\frac{(n+2)^2}{6n(n+1)(2n+1)(4n+5)} \right]^{(n+1)/2} M^{n+1} \right\}^{-(2n^2+2n-1)/(n+2)^2}. \end{aligned} \quad (3.13)$$

For $n = -7/2$, because when $t \rightarrow t_s$, $\hat{t} \rightarrow \mp\infty$, it follows from Eq. (3.12) that the scale factor in the scalar field theory $\hat{a}(\hat{t})$ diverges at infinite time.

Consequently, the ‘finite-time’ Big Rip singularity in $F(R)$ gravity, i.e., in the Jordan frame, becomes the ‘infinite-time’ one in the corresponding scalar field theory obtained through the conformal transformation of the theory of $F(R)$ gravity, namely, in the Einstein frame. This shows the physical difference of late-time cosmological evolutions between the theory of $F(R)$ gravity and the corresponding scalar field theory, which are mathematically equivalent theories.

IV. RELATIONS BETWEEN SCALAR FIELD THEORIES AND THE CORRESPONDING THEORIES OF $F(R)$ GRAVITY

In this section, following the considerations in Refs. [47, 49], we investigate the relations between scalar field theories and the corresponding transformations to $F(R)$ gravity.

The action of scalar field theories in the Einstein frame is given by

$$S_\chi = \int d^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{2\kappa^2} \mp \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{W}(\chi) \right]. \quad (4.1)$$

Here, in the non-phantom phase the sign of the kinetic term is $-$, while in the phantom one that is $+$.

To study the corresponding theories of $F(R)$ gravity, we make the inverse conformal transformation of the action of the scalar field theories (4.1). In the non-phantom phase, we use the inverse conformal transformation [49] in order to vanish the kinetic term of χ

$$\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = e^{\pm\sqrt{2/3}\kappa\chi} \hat{g}_{\mu\nu}. \quad (4.2)$$

As a consequence, the action in the Jordan frame for the non-phantom phase is given by

$$S_{\text{NP}} = \int d^4x \sqrt{-g} \frac{F_{\text{NP}}(R)}{2\kappa^2}, \quad (4.3)$$

$$F_{\text{NP}}(R) \equiv e^{\pm\sqrt{2/3}\kappa\chi(R)} R - 2\kappa^2 e^{\pm 2\sqrt{2/3}\kappa\chi(R)} \tilde{W}(\chi(R)). \quad (4.4)$$

The scalar field χ is just an auxiliary field and can be expressed in terms of the scalar curvature as $\chi = \chi(R)$ by solving the equation of motion of χ :

$$R = e^{\pm\sqrt{2/3}\kappa\chi} \left(4\kappa^2 \tilde{W}(\chi) \pm \sqrt{6}\kappa \frac{d\tilde{W}(\chi)}{d\chi} \right). \quad (4.5)$$

Similarly, in the phantom phase we use the complex conformal transformation [47] in order to vanish the kinetic term of χ

$$\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = e^{\pm i\sqrt{2/3}\kappa\chi} \hat{g}_{\mu\nu}. \quad (4.6)$$

As a result, the action in the Jordan frame for the phantom phase is given by

$$S_P = \int d^4x \sqrt{-g} \frac{F_P(R)}{2\kappa^2}, \quad (4.7)$$

$$F_P(R) \equiv e^{\pm i\sqrt{2/3}\kappa\chi(R)} R - 2\kappa^2 e^{\pm i2\sqrt{2/3}\kappa\chi(R)} \tilde{W}(\chi(R)). \quad (4.8)$$

The equation of motion of χ is given by

$$R = e^{\pm i\sqrt{2/3}\kappa\chi} \left(4\kappa^2 \tilde{W}(\chi) \mp i\sqrt{6}\kappa \frac{d\tilde{W}(\chi)}{d\chi} \right). \quad (4.9)$$

This equation can be solved with respect to χ as $\chi = \chi(R)$.

In general, scalar field theories describing the non-phantom (phantom) phase can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation [47, 49]. We note that the consideration of this section can be applied to not only the model in Eq. (2.18) but also to any other scalar field theories with/without the crossing of the phantom divide: e.g., the case in which the Hubble rate is given by Eq. (B4). In Appendix D, we examine the more detailed relation between the scalar field theories with realizing a crossing of the phantom divide and the corresponding theories of $F(R)$ gravity.

V. MODEL OF $F(R)$ GRAVITY WITH THE TRANSITION FROM THE DE SITTER UNIVERSE TO THE PHANTOM PHASE

In this section, we reconstruct a model of $F(R)$ gravity in which the transition from the de Sitter universe to the phantom phase can occur by using the method explained in Sec. II A.

A. Reconstruction of the viable $F(R)$ gravity

The interesting viable model is proposed in Ref. [36]. It is known that the above model is a very realistic modified gravitational theory that evade solar-system tests, which was

mentioned also in Ref. [50]. As shown in the above reference [36], this model could reproduce the viable cosmic expansion, correctly describing the phases before dark energy epoch. Hence our universe is asymptotically de Sitter space. The form of the model in Ref. [36] and its generalization is presented in Appendix E.

As an example realizing the transition from the de Sitter universe to the phantom phase, we can consider the following form of the Hubble rate:

$$H = g_0 + \frac{g_1}{t_s - t} , \quad (5.1)$$

where g_0 , g_1 and t_s are positive constants. When $t \rightarrow -\infty$, H goes to a constant $H \rightarrow g_0$. Hence the universe is asymptotically de Sitter space. On the other hand, when $t \rightarrow t_s$, the second term on the right-hand side of Eq. (5.1) dominates and H behaves as $H \sim g_1/(t_s - t)$. It follows from $\dot{H} \sim g_1/(t_s - t)^2 > 0$ and Eq. (2.24) that $w_{\text{eff}} < -1$, namely, the universe enters the phantom phase. Then there appears the Big Rip singularity at $t = t_s$. For the case of Eq. (5.1), R is given by

$$R = 6 \left[2g_0^2 + \frac{4g_0g_1}{t_s - t} + \frac{g_1(2g_1 + 1)}{(t_s - t)^2} \right] . \quad (5.2)$$

We consider the case in which the contribution from matter could be neglected. We take into account it later. Eq. (5.1) shows

$$\frac{d\tilde{g}(\phi)}{d\phi} = g_0 + \frac{g_1}{t_s - \phi} , \quad (5.3)$$

where we have taken $\phi = t$. Substituting Eq. (5.3) into Eq. (2.11), we obtain

$$0 = \frac{d^2 P(\phi)}{d\phi^2} - \left(g_0 + \frac{g_1}{t_s - \phi} \right) \frac{dP(\phi)}{d\phi} + \frac{2g_1}{(t_s - \phi)^2} P(\phi) . \quad (5.4)$$

The solution is given by

$$P(z) = C_+ z^\alpha F_K(\alpha, \tilde{\gamma}; z) + C_- z^{1-\tilde{\gamma}} F_K(\alpha - \tilde{\gamma} + 1, 2 - \tilde{\gamma}; z) , \quad (5.5)$$

where

$$z \equiv g_0(\phi - t_s) , \quad \alpha \equiv \frac{1 - g_1 \pm \sqrt{g_1^2 - 10g_1 + 1}}{4} , \quad \tilde{\gamma} \equiv 1 \pm \frac{\sqrt{g_1^2 - 10g_1 + 1}}{2} ,$$

$$F_K(\alpha, \tilde{\gamma}; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1)}{\tilde{\gamma}(\tilde{\gamma}+1) \cdots (\tilde{\gamma}+n-1)} \frac{z^n}{n!} . \quad (5.6)$$

Here, F_K is the Kummer functions (confluent hypergeometric function), and C_+ and C_- are dimensionless constants. Using Eqs. (2.12) and (5.5), we obtain

$$Q(z) = -6g_0^2 \left(1 - \frac{g_1}{z}\right) \left\{ C_+ z^{\alpha-1} \left[(\alpha - g_1 + z) F_K(\alpha, \tilde{\gamma}; z) + \frac{\alpha}{\tilde{\gamma}} z F_K(\alpha + 1, \tilde{\gamma} + 1; z) \right] \right. \\ \left. + C_- z^{-\tilde{\gamma}} \left[(1 - \tilde{\gamma} - g_1 + z) F_K(\alpha - \tilde{\gamma} + 1, 2 - \tilde{\gamma}; z) \right. \right. \\ \left. \left. + \frac{\alpha - \tilde{\gamma} + 1}{2 - \tilde{\gamma}} z F_K(\alpha - \tilde{\gamma} + 2, 3 - \tilde{\gamma}; z) \right] \right\}. \quad (5.7)$$

In Eqs. (5.5) and (5.7), because we take $\phi = t$, $z = g_0(t - t_s)$.

It follows from Eq. (5.2) that when $t \rightarrow -\infty$, namely, in de Sitter phase, R becomes constant as $R \sim 12g_0^2$. On the other hand, when $t \rightarrow t_s$, $R \sim 6g_1(2g_1 + 1)/(t_s - t)^2$. Using this relation and $z = g_0(t - t_s)$, we find

$$z \sim -g_0 \sqrt{\frac{6g_1(2g_1 + 1)}{R}}. \quad (5.8)$$

In the limit of $t \rightarrow t_s$, $|z| \ll 1$ because R diverges. Expanding the Kummer functions in Eqs. (5.5) and (5.7) and taking the first leading order in z , from Eqs. (2.4) and (5.8) we find that the form of $F(R)$ in this limit is approximately expressed as

$$F(R) \approx \frac{R}{(\tilde{\gamma} - 1)(2g_1 + 1)} \left\{ C_+ (\alpha - 1)(\alpha + g_1 + 1) \left[-g_0 \sqrt{6g_1(2g_1 + 1)} \right]^\alpha R^{-\alpha/2} \right. \\ \left. - C_- (\alpha - \tilde{\gamma})(2 - \tilde{\gamma} + g_1) \left[-g_0 \sqrt{6g_1(2g_1 + 1)} \right]^{1-\tilde{\gamma}} R^{-(1-\tilde{\gamma})/2} \right\}. \quad (5.9)$$

Next, we study the case in which there exists the matter. We consider the cold dark matter with $w = 0$. We numerically solve Eq. (2.11) with the cold dark matter. To execute this, for simplicity, we set $t_s g_0 = g_1$ in Eq. (5.1). We show the behavior of $F(\tilde{R})/(2\kappa^2)$ in Fig. 2. From Fig. 2, we see that $F(\tilde{R})$ increases in terms of \tilde{R} . The detailed explanation of the numerical calculations is given in Appendix F.

B. Addition of a non-local term to the viable modified gravity models

It seems very difficult in the framework of $F(R)$ gravity to construct a model generating the transition from de Sitter space to the phantom phase in the viable model of the previous sub-section. One may add any term, which is a function of R , to such models. The term should be small in the present universe and may be small even in the past universe, where

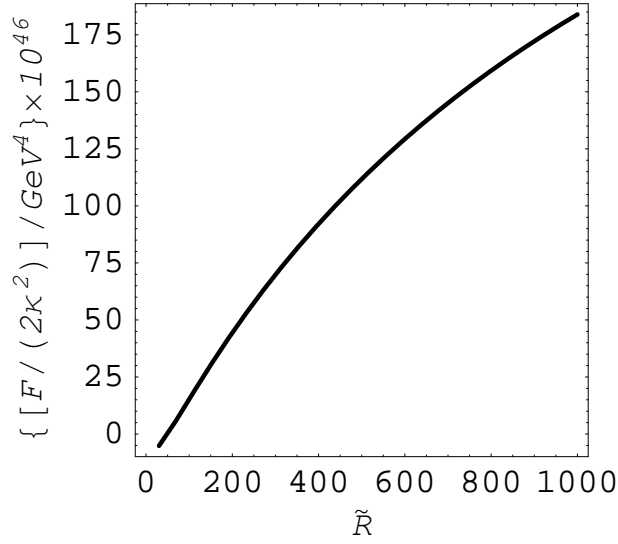


FIG. 2: Behavior of $F(\tilde{R})/(2\kappa^2)$ as a function of \tilde{R} . Legend is the same as Fig. 3.

the curvature could be large. Hence the term should dominate only at small curvature, that is, smaller one than the present curvature. In the asymptotically-de Sitter model above, however, the present universe is asymptotic de Sitter space, where the curvature is (almost) constant. Thus, the curvature cannot become smaller than the order of the curvature in the present universe and the added term never dominates.

Let us add a non-local term to any model from Ref. [37] by using a proper function K :

$$\delta f = K(-\square^{-1}R) R^m, \quad (5.10)$$

where m is a positive constant. In the de Sitter universe where the curvature and the Hubble rate are constant $R = R_0$, $H = H_0$, we find

$$\square^{-1}R = -\frac{R_0}{3H_0}t + c_4 e^{-3H_0 t} + c_5, \quad (5.11)$$

where c_4 and c_5 are constants of the integration. For large t , we find $-\square^{-1}R \sim [R_0/(3H_0)]t$. If we choose K to be a slowly increasing function, δf dominates in the future. If K is slowly varying function and could be regarded to be a constant, the total $F(R)$ behaves as $F(R) \sim R^m$, which gives

$$H = \frac{\tilde{h}_0}{t}, \quad \tilde{h}_0 = -\frac{(m-1)(2m-1)}{m-2}, \quad (5.12)$$

when $\tilde{h}_0 > 0$ or

$$H = \frac{-\tilde{h}_0}{t_s - t}, \quad \tilde{h}_0 = -\frac{(m-1)(2m-1)}{m-2}, \quad (5.13)$$

when $\tilde{h}_0 < 0$. Here, \tilde{h}_0 is a constant.

Alternatively, the addition of an extra scalar field may bring the evolution to the phantom era. This is because such terms can become larger at the constant curvature and hence may induce such a crossing in realistic models.

At present, it is not so clear if such non-local models could be variable or not due to technical problems, but this model has a possibility to explain the complicated cosmic expansion, especially the coincidence problem. We now consider the models including such non-local terms to show the generality of our method.

We remark that as shown in Ref. [26], such a theory may successfully pass the solar system tests, and that as demonstrated in Ref. [25], such non-local models may correctly reproduce the whole expansion history of the universe expansion, at least in their equivalent scalar-tensor form.

VI. STABILITY UNDER A QUANTUM CORRECTION

In this section, we examine the stability for the obtained solutions of the crossing of the phantom divide under a quantum correction of massless conformally-invariant fields. It is convenient to do it by taking account of conformal anomaly induced effective pressure and energy-density. Note that we do not discuss the quantum regime of modified gravity itself because it is relevant at strong curvature (near to the Planck scale) where the form of modified gravity may be quite different from the one at the late universe.

Quantum effects produce the conformal anomaly:

$$T_A = b \left(F + \frac{2}{3} \square R \right) + b' G + b'' \square R, \quad (6.1)$$

Here F : the square of 4d Weyl tensor, G : Gauss-Bonnet invariant, which are given as

$$F = \frac{1}{3} R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}, \quad G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}. \quad (6.2)$$

In the FRW background (2.6), we find

$$F = 0, \quad G = 24 \left(\dot{H}H^2 + H^4 \right). \quad (6.3)$$

For N real scalar, $N_{1/2}$ Dirac spinor, N_1 vector fields, N_2 ($= 0$ or 1) gravitons and N_{HD} higher derivative conformal scalars,

$$\begin{aligned} b &= \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2} , \\ b' &= -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\text{HD}}}{360(4\pi)^2} . \end{aligned} \quad (6.4)$$

b'' can be arbitrary and we may choose, for example, $b'' = -2b/3$ or $b'' = 0$. If we assume T_A can be given by the effective energy density ρ_A and pressure p_A from the conformal anomaly as

$$T_A = -\rho_A + 3p_A , \quad (6.5)$$

and ρ_A and p_A satisfy the conservation law,

$$\dot{\rho}_A + 3H(\rho_A + p_A) = 0 , \quad (6.6)$$

we find

$$\rho_A = -\frac{1}{a^4} \int dt a^4 H T_A , \quad p_A = -\frac{1}{3a^4} \int dt a^4 H T_A + \frac{T_A}{3} . \quad (6.7)$$

On the other hand, Eqs. (C6) and (C7) give the effective energy density ρ_F and pressure p_F from $f(R) = F(R) - R$ term:

$$\begin{aligned} \kappa^2 \rho_F &= -\frac{1}{2} (F(R) - R) + 3 \left(H^2 + \dot{H} \right) (F'(R) - 1) - 18 \left(4H^2 \dot{H} + H \ddot{H} \right) F''(R) , \\ \kappa^2 p_F &= \frac{1}{2} (F(R) - R) - \left(3H^2 + \dot{H} \right) (F'(R) - 1) \\ &\quad + 6 \left(8H^2 \dot{H} + 4\dot{H}^2 + 6H \ddot{H} + \ddot{H} \right) F''(R) + 36 \left(4H \dot{H} + \ddot{H} \right)^2 F'''(R) . \end{aligned} \quad (6.8)$$

We now investigate the magnitude of ρ_A , p_A , ρ_F , and p_F when the phantom crossing occurs, when $\dot{H} = 0$. We assume the magnitude of the Hubble rate H could be the order of the present Hubble constant H_0 :

$$H \sim H_0 \sim 10^{-33} \text{ eV} . \quad (6.9)$$

Expressions on (6.7) tells that we may assume $\rho_A \sim p_A \sim T_A$. Then we find

$$\rho_A \sim p_A \sim C H_0^4 . \quad (6.10)$$

Here C is a dimensionless constant coming from b , b' , b'' , and numerical constants and therefore $C \sim 10^{2\sim 3}$. On the other hand, expressions in (6.8) tell that we may assume

$\rho_F \sim p_F \sim f(R)/\kappa^2$. Since $f(R)$ plays the role of the effective cosmological constant, we also assume $f(R) \sim H_0^2$ and we find

$$\rho_F \sim p_F \sim \frac{H_0^2}{\kappa^2} . \quad (6.11)$$

Since $1/\kappa \sim 10^{28}$ eV, we find

$$|\rho_F| \gg |\rho_A| , \quad |p_F| \gg |p_A| . \quad (6.12)$$

Therefore the quantum correction could be small when the phantom crossing occurs and the obtained solutions of the phantom crossing in this paper could be stable under the quantum correction. We should note that the quantum correction becomes important near the Big Rip singularity, where the curvature becomes very large.

VII. CONCLUSION

In the present paper, we have considered a crossing of the phantom divide in modified gravity. We have reconstructed an explicit model of modified gravity in which a crossing of the phantom divide can occur by using the reconstruction method proposed in Ref. [33]. As a result, we have shown that the (finite-time) Big Rip singularity appears in the reconstructed model of modified gravity (i.e., in the Jordan frame), whereas that in the corresponding scalar field theory obtained through the conformal transformation (i.e., in the Einstein frame) the singularity becomes the infinite-time one. Furthermore, we have investigated the relations between scalar field theories with realizing a crossing of the phantom divide and the corresponding modified gravitational ones by using the inverse conformal transformation of scalar field theories. It has been demonstrated that the scalar field theories describing the non-phantom phase (phantom one with the Big Rip singularity) can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation. Moreover, taking into account the fact that in the viable models [36, 37, 38], which are very realistic modified gravities that evade solar-system tests, our universe is asymptotically de Sitter space, we have also proposed a model of modified gravity in which the transition from the de Sitter universe to the phantom phase can occur. We have found that to construct a viable model generating the transition from de Sitter space to the phantom phase, additional non-local term or almost equivalent scalar field is necessary. It would be

interesting to reconsider this problem in the presence of ideal fluid matter. In addition, we have examined the stability for the obtained solutions of the crossing of the phantom divide under a quantum correction coming from conformal anomaly.

The study of the future evolution of the universe as discussed in this paper may be important to understand whether our universe evolves eternally or it will enter into the finite-time singularity. Additionally, this may shed extra light to some specific properties of different dark energy models and may help in selecting correct descriptions for the dark side of the universe.

The originality of this work is to reconstruct an explicit model of modified gravity in which a crossing of the phantom divide can be realized. This is the point beyond the already existing literature. We have demonstrated that in principle the crossing of the phantom divide can occur at the present time or in the near future in the framework of modified gravity without introducing any extra scalar components with the wrong kinetic sign such as a phantom. This corresponds to the proof that the crossing of the phantom divide is possible also for modified gravity theories, which seem to be much less pathologic than the usual phantom scalar models of dark energy, similar to the scalar field theories in the framework of general relativity. The demonstration in this work can be regarded as a significant step to construct a more realistic model of modified gravity to correctly describe the expansion history of the universe.

Acknowledgments

The work by K.B. and C.Q.G. is supported in part by the National Science Council of R.O.C. under: Grant #s: NSC-95-2112-M-007-059-MY3 and National Tsing Hua University under Grant #: 97N2309F1 (NTHU), that by S.D.O. was supported in part by MEC (Spain) projects FIS2006-02842 and PIE2007-50I023, RFBR grant 06-01-00609 and LRSS project N.2553.2008.2, and that by S.N. is supported in part by the Ministry of Education, Science, Sports and Culture of Japan under grant no.18549001 and Global COE Program of Nagoya University provided by the Japan Society for the Promotion of Science (G07).

APPENDIX A: NOTE ON THE RECONSTRUCTION METHOD

In this appendix, we note the following point on the reconstruction method explained in Sec. II A.

In the action (2.2), if we redefine the auxiliary scalar field ϕ by $\phi = \Phi(\varphi)$ by using a proper function Φ , and define $\tilde{P}(\varphi) \equiv P(\Phi(\varphi))$ and $\tilde{Q}(\varphi) \equiv Q(\Phi(\varphi))$, the action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} [\tilde{P}(\varphi)R + \tilde{Q}(\varphi)] + \mathcal{L}_{\text{matter}} \right\} \quad (\text{A1})$$

is equivalent to the action (2.2) because this gives identical $F(R)$ gravity. This can be explicitly confirmed as follows. First we should note that Eq. (2.3) is modified as

$$0 = \frac{d\tilde{P}(\varphi)}{d\varphi}R + \frac{d\tilde{Q}(\varphi)}{d\varphi} = \left\{ \frac{dP(\Phi(\varphi))}{d\Phi}R + \frac{dQ(\Phi(\varphi))}{d\Phi} \right\} \frac{d\Phi}{d\varphi}. \quad (\text{A2})$$

Then we can solve φ with respect to R by

$$\varphi = \varphi(R) = \Phi^{-1}(\phi(R)). \quad (\text{A3})$$

Here, φ is the inverse function of Φ . Hence the obtained $\tilde{F}(R)$, corresponding to (2.4), is

$$\begin{aligned} \tilde{F}(R) &\equiv \tilde{P}(\varphi(R))R + \tilde{Q}(\varphi(R)) = P(\Phi(\Phi^{-1}(\phi(R))))R + Q(\Phi(\Phi^{-1}(\phi(R)))) \\ &= P(\phi(R))R + Q(\phi(R)) = F(R). \end{aligned} \quad (\text{A4})$$

Thus the obtained $F(R)$ could be identical. Consequently, there are always ambiguities for the choice in ϕ like a gauge symmetry. In the FRW universe, we now assume that we have solved the $F(R)$ -gravity theory and obtained R as a function of time t as $R = R(t)$. Then ϕ can be expressed as a function of t , $\phi = \tilde{\phi}(t)$, by a proper function $\tilde{\phi}$. If we redefine a scalar field by $\varphi = \tilde{\phi}^{-1}(\phi)$ by using the inverse function $\tilde{\phi}^{-1}$ of $\tilde{\phi}$, we obtain $\varphi = t$. Hence we can always, at least locally, identify ϕ with time t , $\phi = t$, which can be interpreted as a gauge condition corresponding to the reparameterization of $\phi = \phi(\varphi)$.

There could be several cases that we cannot construct $F(R)$. One possibility is that the differential equation (2.11) has no consistent solution. Another possibility could be the case that the algebraic equation (2.3) has no solution for obtained P and Q (for example, P and/or Q is a constant).

APPENDIX B: RECONSTRUCTION OF AN EXPLICIT MODEL

In this appendix, we demonstrate that Eq. (2.13) can be a solution of Eq. (2.11) without matter.

We start with Eq. (2.11) without matter:

$$0 = \frac{d^2 P(\phi)}{d\phi^2} - \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} + 2 \frac{d^2 \tilde{g}(\phi)}{d\phi^2} P(\phi) . \quad (\text{B1})$$

By redefining $P(\phi)$ as Eq. (2.13), Eq. (B1) is rewritten to

$$\frac{1}{\tilde{p}(\phi)} \frac{d^2 \tilde{p}(\phi)}{d\phi^2} = 25 e^{\tilde{g}(\phi)/10} \frac{d^2 (e^{-\tilde{g}(\phi)/10})}{d\phi^2} . \quad (\text{B2})$$

We now consider the model Eq. (2.14). In this case, Eq. (B2) is reduced to

$$\frac{1}{\tilde{p}(\phi)} \frac{d^2 \tilde{p}(\phi)}{d\phi^2} = \frac{25\gamma(\gamma+1)}{\phi^2} , \quad (\text{B3})$$

which can be solved as Eq. (2.15).

We mention the following point about the form of $\tilde{g}(\phi)$ in Eq. (2.14). From Eq. (2.19), we see that as the universe evolves, the sign of \dot{H} has to change in time so that a crossing of the phantom divide can occur. To realize such a behavior of H , there must exist (at least) two terms of ϕ in the brackets $[\]$ of the logarithmic function on the right-hand side of Eq. (2.14). (Incidentally, the reason why we select the coefficient ‘10’ on the right-hand side of Eq. (2.20) is to obtain the solution analytically.) As another form of the Hubble rate realizing a crossing of the phantom divide, there is the following model [15]:

$$H = h_0 \left(\frac{1}{t} + \frac{1}{t_s - t} \right) , \quad (\text{B4})$$

where h_0 is a positive constant. It has been shown that this cosmology can be constructed in terms of multiple scalar field theories [15]. This form also consists of two terms in t . In this model, $\dot{H} = h_0 (2t - t_s) t_s / [t^2 (t_s - t)^2]$. It follows from Eq. (2.19) that when $t < t_s/2$, $\dot{H} < 0$ and hence the universe is in non-phantom phase ($w_{\text{eff}} > -1$), but that when $t > t_s/2$, $\dot{H} > 0$ and thus the universe is in phantom phase ($w_{\text{eff}} < -1$). It is not hard to formulate the explicit model of modified gravity with the above type of the crossing of the phantom divide.

APPENDIX C: DERIVATION OF THE CORRESPONDING SCALAR FIELD THEORY

In this appendix, we derive the expression of the action (3.4).

It follows from the action (3.1) that the equation of motion of the other auxiliary field ζ is given by

$$\xi = F'(\zeta), \quad (\text{C1})$$

where the prime denotes differentiation with respect to ζ . Substituting Eq. (C1) into Eq. (3.1) and eliminating ξ from Eq. (3.1), we find

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} (F'(\zeta)R + F(\zeta) - F'(\zeta)\zeta) + \mathcal{L}_{\text{matter}} \right]. \quad (\text{C2})$$

We make the conformal transformation (3.2) with Eq. (3.3) of the action (C2). Consequently, the action in the Einstein frame is given by [39, 48]

$$S_E = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2\kappa^2} \left(\hat{R} - \frac{3}{2} \hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right) + e^{-2\sigma} \mathcal{L}_{\text{matter}} \right], \quad (\text{C3})$$

where

$$V(\sigma) = e^{-\sigma} \zeta(\sigma) - e^{-2\sigma} F(\zeta(\sigma)) = \frac{\zeta}{F'(\zeta)} - \frac{F(\zeta)}{(F'(\zeta))^2}, \quad (\text{C4})$$

and \hat{g} is the determinant of $\hat{g}^{\mu\nu}$. In deriving Eqs. (C3) and (C4), we have used Eq. (3.3). In addition, $\zeta(\sigma)$ in Eq. (C4) is obtained by solving Eq. (3.3) with respect to ζ as $\zeta = \zeta(\varphi)$. By defining φ as $\varphi \equiv \sqrt{3/2}\sigma/\kappa$, the action (C3) is reduced to the form of the canonical scalar field theory (3.4).

From the action (2.1), we find that the gravitational field equation is given by

$$F'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(R) + g_{\mu\nu}\square F'(R) - \nabla_\mu \nabla_\nu F'(R) = \kappa^2 T_{\mu\nu}^{(\text{matter})}. \quad (\text{C5})$$

When there is no matter, in the FRW background (2.6) the $(\mu, \nu) = (0, 0)$ component and the trace part of the $(\mu, \nu) = (i, j)$ component of Eq. (C5), where i and j run from 1 to 3, are given by

$$3H^2 = -\frac{1}{2}(F(R) - R) + 3\left(H^2 + \dot{H}\right)(F'(R) - 1) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F''(R), \quad (\text{C6})$$

and

$$- \left(2\dot{H} + 3H^2 \right) = \frac{1}{2} (F(R) - R) - \left(3H^2 + \dot{H} \right) (F'(R) - 1) \\ + 6 \left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \ddot{H} \right) F''(R) + 36 \left(4H\dot{H} + \ddot{H} \right)^2 F'''(R), \quad (\text{C7})$$

respectively. Using Eqs. (C6) and (C7), it follows

$$2\dot{H}F'(R) + 6 \left(-4H^2\dot{H} + 4\dot{H}^2 + 3H\ddot{H} + \ddot{H} \right) F''(R) + 36 \left(4H\dot{H} + \ddot{H} \right)^2 F'''(R) = 0. \quad (\text{C8})$$

APPENDIX D: CORRESPONDENCE BETWEEN THE SCALAR FIELD THEORIES AND $F(R)$ GRAVITIES

In this appendix, we explore the scalar field theories with realizing a crossing of the phantom divide in the Einstein frame and consider the behavior of the corresponding theories of $F(R)$ gravity.

1. Scalar field theories

The action of scalar field theories in the Einstein frame is given by

$$S_\Phi = \int d^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \omega(\Phi) \hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - W(\Phi) \right], \quad (\text{D1})$$

where $\omega(\Phi)$ is a functions of the scalar field Φ and $W(\Phi)$ is the potential of Φ .

In the FRW background (2.6), the Einstein equations are given by

$$\frac{3}{\kappa^2} H^2 = \rho_\Phi, \quad -\frac{2}{\kappa^2} \dot{H} = p_\Phi + \rho_\Phi, \quad (\text{D2})$$

where the energy density ρ_Φ of the scalar field Φ and the pressure p_Φ of it are given by

$$\rho_\Phi = \frac{1}{2} \omega(\Phi) \dot{\Phi}^2 + W(\Phi), \quad p_\Phi = \frac{1}{2} \omega(\Phi) \dot{\Phi}^2 - W(\Phi), \quad (\text{D3})$$

respectively. Using equations in (D2) and (D3), we obtain

$$\omega(\Phi) \dot{\Phi}^2 = -\frac{2}{\kappa^2} \dot{H}, \quad (\text{D4})$$

$$W(\Phi) = \frac{1}{\kappa^2} \left(3H^2 + \dot{H} \right). \quad (\text{D5})$$

It is the interesting case that $\omega(\Phi)$ and $W(\Phi)$ are defined in terms of a single function $I(\Phi)$ as [49]

$$\omega(\Phi) = -\frac{2}{\kappa^2} \frac{dI(\Phi)}{d\Phi}, \quad (\text{D6})$$

$$W(\Phi) = \frac{1}{\kappa^2} \left(3I^2(\Phi) + \frac{dI(\Phi)}{d\Phi} \right). \quad (\text{D7})$$

Thus we can find the solutions

$$\Phi = t, \quad H = I(t). \quad (\text{D8})$$

In what follows, we consider the case in which these solutions are satisfied.

If we define a new scalar field χ as

$$\chi \equiv \int d\Phi \sqrt{|\omega(\Phi)|}, \quad (\text{D9})$$

the action (D1) can be rewritten to the form in Eq. (4.1), where the sign in front of the kinetic term depends on that of $\omega(\Phi)$. If the sign of $\omega(\Phi)$ is positive (negative), that of the kinetic term is $-$ ($+$). In the non-phantom phase, the sign of the kinetic term is always $-$, and in the phantom one it is always $+$. In principle, it follows from Eq. (D9) that Φ can be solved with respect to χ as $\Phi = \Phi(\chi)$. Hence, the potential $\tilde{W}(\chi)$ is given by $\tilde{W}(\chi) = W(\Phi(\chi))$.

In the case of the model explained in Sec. II B, it follows from Eq. (2.18) that $I(\Phi)$ is given by

$$I(\Phi) = \left(\frac{10}{\Phi} \right) \left[\frac{\gamma + (\gamma + 1) \left(\frac{\Phi}{t_s} \right)^{2\gamma+1}}{1 - \left(\frac{\Phi}{t_s} \right)^{2\gamma+1}} \right]. \quad (\text{D10})$$

From the solutions in (D8), we find that Eq. (D10) gives

$$H = \left(\frac{10}{t} \right) \left[\frac{\gamma + (\gamma + 1) \left(\frac{t}{t_s} \right)^{2\gamma+1}}{1 - \left(\frac{t}{t_s} \right)^{2\gamma+1}} \right]. \quad (\text{D11})$$

In deriving the expressions in (D10) and (D11), we have used Eq. (2.17). In this case, from

Eqs. (D6), (D7) and (D9) we find

$$\omega(\Phi) = \frac{20 \left[\gamma - 4\gamma(\gamma+1) \left(\frac{\Phi}{t_s} \right)^{2\gamma+1} - (\gamma+1) \left(\frac{\Phi}{t_s} \right)^{2(2\gamma+1)} \right]}{\kappa^2 \Phi^2 \left[1 - \left(\frac{\Phi}{t_s} \right)^{2\gamma+1} \right]^2}, \quad (\text{D12})$$

$$\chi = \frac{\sqrt{20}}{\kappa} \int d\Phi \frac{\sqrt{\left| \gamma - 4\gamma(\gamma+1) \left(\frac{\Phi}{t_s} \right)^{2\gamma+1} - (\gamma+1) \left(\frac{\Phi}{t_s} \right)^{2(2\gamma+1)} \right|}}{\Phi \left[1 - \left(\frac{\Phi}{t_s} \right)^{2\gamma+1} \right]}, \quad (\text{D13})$$

$$\begin{aligned} \tilde{W}(\chi) &= \frac{10}{\kappa^2 \Phi^2(\chi) \left[1 - \left(\frac{\Phi(\chi)}{t_s} \right)^{2\gamma+1} \right]^2} \\ &\times \left[\gamma(30\gamma-1) + 64\gamma(\gamma+1) \left(\frac{\Phi(\chi)}{t_s} \right)^{2\gamma+1} + (\gamma+1)(30\gamma+31) \left(\frac{\Phi(\chi)}{t_s} \right)^{2(2\gamma+1)} \right]. \end{aligned} \quad (\text{D14})$$

As shown in Sec. II B, when $t < t_c$, $\dot{H} < 0$ (non-phantom phase) and it follows from Eq. (D4) that $\omega > 0$. In the non-phantom phase, the sign of the kinetic term in the action (4.1) is $-$. On the other hand, when $t > t_c$, $\dot{H} > 0$ (phantom phase) and from Eq. (D4) we see that $\omega < 0$. In the phantom phase, the sign of the kinetic term in the action (4.1) is $+$. When $t = t_c$, $\omega = 0$ and the transition from non-phantom phase to phantom one occurs [41].

2. Corresponding theories of $F(R)$ gravity

We investigate the behavior of the modified gravity (4.8) in the limit of $t \rightarrow t_s$ for the case in which the potential $\tilde{W}(\chi(R))$ is given by Eq. (D14). If the scalar field χ is real, it follows from Eq. (4.9) that the scalar curvature R is not always real. In order for R to be real, the following condition should be satisfied [47]:

$$e^{i\sqrt{2/3}\kappa\chi} \left(4\kappa^2 \tilde{W}(\chi) - i\sqrt{6}\kappa \frac{d\tilde{W}(\chi)}{d\chi} \right) = e^{-i\sqrt{2/3}\kappa\chi} \left(4\kappa^2 \tilde{W}(\chi) + i\sqrt{6}\kappa \frac{d\tilde{W}(\chi)}{d\chi} \right). \quad (\text{D15})$$

This condition is reduced to

$$\frac{1}{\tilde{W}(\chi)} \frac{d\tilde{W}(\chi)}{d\chi} = 2\sqrt{\frac{2}{3}}\kappa \tan \sqrt{\frac{2}{3}}\kappa\chi. \quad (\text{D16})$$

Except for the form of $\tilde{W}(\chi)$ satisfying Eq. (D16), R is complex if χ is real. The form of $\tilde{W}(\chi)$ in Eq. (D14) cannot satisfy Eq. (D16). This applies to the case in which the Hubble

rate is given by Eq. (B4). If the scalar field χ is pure imaginary and expressed as $\chi = i\eta$, where η is a real scalar field, and the potential $\tilde{W}(\chi)$ contains only even power of χ , Eq. (4.9) is rewritten to

$$R = e^{\mp\sqrt{2/3}\kappa\eta} \left[4\kappa^2 \tilde{W}(-\eta^2) \pm 2\sqrt{6}\kappa \frac{d\tilde{W}(-\eta^2)}{d(-\eta^2)} \right], \quad (\text{D17})$$

which tells that R is real. In fact, however, that the action (4.1) with the $+$ sign of the kinetic term is reduced to [47]

$$S_\eta = \int d^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \eta \partial_\nu \eta - \tilde{W}(-\eta^2) \right]. \quad (\text{D18})$$

This action corresponds to the non-phantom (canonical) theory. As a consequence, if the scalar field theory describes the phantom phase with the Big Rip singularity, the corresponding theory of $F(R)$ gravity is usually complex. (Note that counterexample is known [47], but even in this case, when $t \rightarrow t_s$, the modified gravity becomes complex.) When $t \rightarrow t_s$, from Eq. (D14) $\tilde{W}(\chi)$ diverges and hence $F_P(R)$ also diverges.

In the Einstein frame, the scalar field couples with matter and therefore, the frame could be unphysical. The coupling changes the scale of time interval. We usually measure the time by using electromagnetism. In the limit of neglecting the local gravity, the measured time corresponds to the cosmological time in the Jordan frame but not in the Einstein frame.

APPENDIX E: EXAMPLES OF VIABLE $F(R)$ GRAVITY MODELS

In this appendix, we show examples of viable $F(R)$ gravity models.

The modified part in $F(R)$ in the action (2.1) can be separated as

$$F(R) = R + f(R), \quad (\text{E1})$$

where $f(R)$ is an arbitrary function of the scalar curvature R . The interesting viable model proposed in Ref. [36] is given by

$$f(R) = f_{\text{HS}}(R) \equiv -\frac{\bar{M}^2 c_2 (R/\bar{M}^2)^p}{c_3 (R/\bar{M}^2)^p + 1} = -\frac{\bar{M}^2 c_2}{c_3} + \frac{\bar{M}^2 c_2 / c_3}{c_3 (R/\bar{M}^2)^p + 1}, \quad (\text{E2})$$

where c_2 and c_3 are dimensionless constants, p is a positive constant, and \bar{M} denotes a mass scale. In this model, when $R/\bar{M}^2 \rightarrow \infty$, $f_{\text{HS}}(R) \sim -\bar{M}^2 c_2 / c_3 = \text{const.}$

The generalizations of the above model which admits the unification of early-time inflation with late-time acceleration (with intermediate radiation/matter dominance) have been proposed in Ref. [37]. The example of such a theory is given by

$$f(R) = -\alpha_0 \left(\tanh \left(\frac{b_0 (R - R_0)}{2} \right) + \tanh \left(\frac{b_0 R_0}{2} \right) \right) - \alpha_I \left(\tanh \left(\frac{b_I (R - R_I)}{2} \right) + \tanh \left(\frac{b_I R_I}{2} \right) \right), \quad (\text{E3})$$

where α_0 , α_I , b_0 and b_I are constant, and R_0 and R_I are constant scalar curvatures. We note that such a theory may correctly describe the whole expansion history of the universe qualitatively: inflation, radiation/matter dominance and dark energy, as it has been explained in Ref. [39].

APPENDIX F: NUMERICAL CALCULATIONS WITH THE MATTER

In this appendix, for the case Eq. (5.1) with $t_s g_0 = g_1$, we explain the numerical calculations of Eq. (2.11) with the matter in detail.

We define $\tilde{g}_0 \equiv t_s g_0$ and $Y \equiv 1 - X$ with $X \equiv t/t_s$. From Eqs. (2.10) and (5.1) with $t_s g_0 = g_1$, $H = d\tilde{g}(t)/(dt)$ and $R = 6(\dot{H} + 2H^2)$, we obtain

$$\tilde{g}(t) = -\tilde{g}_0 (Y + \log Y), \quad (\text{F1})$$

$$a(t) = \left[\left(1 - \frac{1}{\alpha} \right) \frac{1}{Y} \right]^{\tilde{g}_0} \exp \left[\tilde{g}_0 \left(1 - \frac{1}{\alpha} - Y \right) \right], \quad (\text{F2})$$

$$\tilde{R} = t_s^2 R = 6\tilde{g}_0 \left(2\tilde{g}_0 + \frac{4\tilde{g}_0}{Y} + \frac{2\tilde{g}_0 + 1}{Y^2} \right), \quad (\text{F3})$$

where we have taken $\bar{a} = (1 - 1/\alpha)^{\tilde{g}_0} \exp[\tilde{g}_0(1 - 1/\alpha)]$ so that the present value of the scale factor should be unity.

The solution of Eq. (F3) with respect to Y is given by

$$Y(\tilde{R}) = \frac{1}{2\tilde{g}_0^2 - \tilde{R}/6} \left[-2\tilde{g}_0^2 \pm \sqrt{-2\tilde{g}_0^3 + \frac{\tilde{g}_0(2\tilde{g}_0 + 1)\tilde{R}}{6}} \right]. \quad (\text{F4})$$

In what follows, we use the lower sign in Eq. (F4).

For simplicity, we consider the case in which there exists a matter with a constant EoS

parameter $w = p/\rho$. In this case, by using Eq. (F4), Eqs. (2.11) and (2.12) are rewritten to

$$\begin{aligned} & [12\tilde{g}_0(2\tilde{g}_0Y + 2\tilde{g}_0 + 1)]^2 \frac{d^2P(\tilde{R})}{d^2\tilde{R}} \\ & + \tilde{g}_0Y^2 \left\{ 12[4\tilde{g}_0Y + 3(2\tilde{g}_0 + 1)] + Y^3(1 + Y) \right\} \frac{dP(\tilde{R})}{d\tilde{R}} + 2\tilde{g}_0Y^4P(\tilde{R}) \\ & + Y^6t_s^2\kappa^2(1 + w)\bar{\rho} \left\{ \left[\left(1 - \frac{1}{\alpha}\right) \frac{1}{Y} \right]^{\tilde{g}_0} \exp \left[\tilde{g}_0 \left(1 - \frac{1}{\alpha} - Y\right) \right] \right\}^{-3(1+w)} = 0 \end{aligned} \quad (\text{F5})$$

and

$$\begin{aligned} t_s^2Q(\tilde{R}) = & -6\tilde{g}_0^2 \left(\frac{1+Y}{Y} \right)^2 P(\tilde{R}) - 72\tilde{g}_0^2 \left(\frac{1+Y}{Y^3} \right) \left(2\tilde{g}_0 + \frac{2\tilde{g}_0 + 1}{Y} \right) \frac{dP(\tilde{R})}{d\tilde{R}} \\ & + 2t_s^2\kappa^2\bar{\rho} \left\{ \left[\left(1 - \frac{1}{\alpha}\right) \frac{1}{Y} \right]^{\tilde{g}_0} \exp \left[\tilde{g}_0 \left(1 - \frac{1}{\alpha} - Y\right) \right] \right\}^{-3(1+w)}, \end{aligned} \quad (\text{F6})$$

respectively. Here, $\bar{\rho}$ corresponds to the present energy density of the matter. In particular, we use the present value of the cold dark matter with $w = 0$ for $\bar{\rho}$, i.e., $\bar{\rho} = 0.233\rho_c$ [2], where $\rho_c = 3H_0^2/(8\pi G) = 3.97 \times 10^{-47}\text{GeV}^4$ is the critical energy density. From Eq. (2.4) and $\tilde{R} = t_s^2R$, we have

$$\frac{F(\tilde{R})}{2\kappa^2} = \frac{1}{2\kappa^2t_s^2} \left(P(\tilde{R})\tilde{R} + t_s^2Q(\tilde{R}) \right). \quad (\text{F7})$$

To examine $F(\tilde{R})$, we numerically solve Eqs. (F5)–(F7).

In Fig. 3, we depict $P(\tilde{R})$ and $Q(\tilde{R})/\nu^2$ as functions of \tilde{R} . The range of \tilde{R} is given by $30 \leq \tilde{R} \leq 1000$, corresponding to $0 < X = t/t_s < 1$. Here, we have taken the initial conditions as $P(\tilde{R} = 30) = 1.0$ and $dP(\tilde{R} = 30)/(d\tilde{R}) = 1.0$. By using Eq. (F7), we show the behavior of $F(\tilde{R})/(2\kappa^2)$ in Fig. 2.

-
- [1] D. N. Spergel *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **148**, 175 (2003); H. V. Peiris *et al.* [WMAP Collaboration], *ibid.* **148**, 213 (2003); D. N. Spergel *et al.* [WMAP Collaboration], *ibid.* **170**, 377 (2007).
 - [2] E. Komatsu *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **180**, 330 (2009).
 - [3] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], *Astrophys. J.* **517**, 565 (1999); A. G. Riess *et al.* [Supernova Search Team Collaboration], *Astron. J.* **116**, 1009 (1998); P. Astier *et al.* [The SNLS Collaboration], *Astron. Astrophys.* **447**, 31 (2006); A. G. Riess *et al.*, *Astrophys. J.* **659**, 98 (2007).

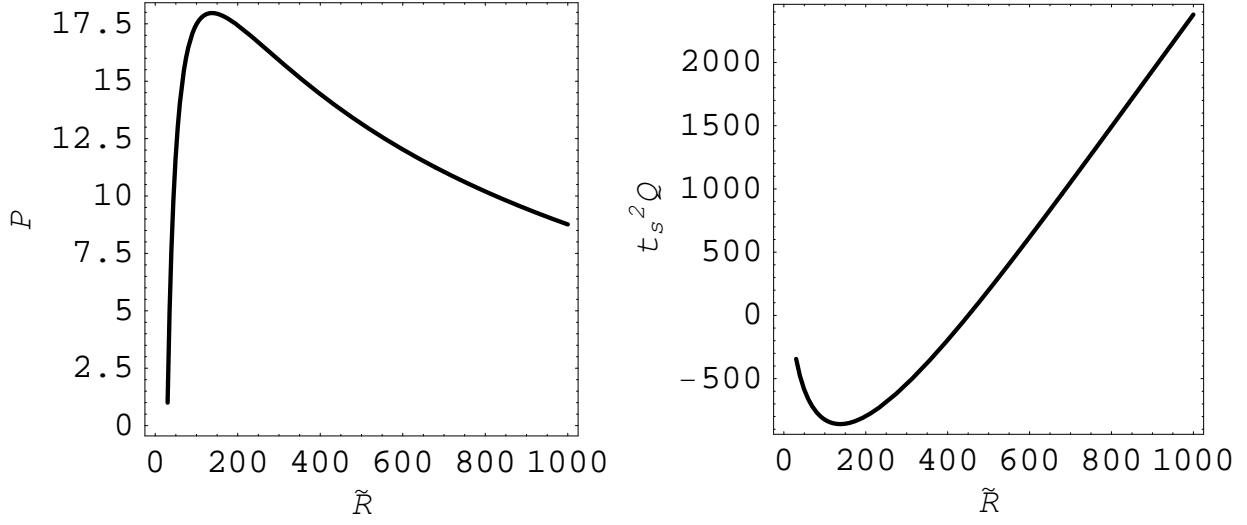


FIG. 3: $P(\tilde{R})$ and $Q(\tilde{R})/\nu^2$ as functions of \tilde{R} for $t_s = 2t_0$ and $\bar{\rho} = 0.233\rho_c$.

- [4] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003); V. Sahni, AIP Conf. Proc. **782**, 166 (2005) [J. Phys. Conf. Ser. **31**, 115 (2006)].
- [5] T. Padmanabhan, Phys. Rept. **380**, 235 (2003).
- [6] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006).
- [7] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007) [arXiv:hep-th/0601213].
- [8] S. Nojiri and S. D. Odintsov, arXiv:0801.4843 [astro-ph]; arXiv:0807.0685 [hep-th]; T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr-qc]; F. S. N. Lobo, arXiv:0807.1640 [gr-qc]; S. Capozziello and M. Francaviglia, Gen. Rel. Grav. **40**, 357 (2008).
- [9] U. Alam, V. Sahni and A. A. Starobinsky, JCAP **0406**, 008 (2004); S. Nesseris and L. Perivolaropoulos, *ibid.* **0701**, 018 (2007); P. U. Wu and H. W. Yu, Phys. Lett. B **643**, 315 (2006); H. K. Jassal, J. S. Bagla and T. Padmanabhan, arXiv:astro-ph/0601389.
- [10] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003) [arXiv:hep-th/0303117].
- [11] L. Perivolaropoulos, JCAP **0510**, 001 (2005); K. Nozari and S. D. Sadatian, Eur. Phys. J. C **58**, 499 (2008).
- [12] S. Nojiri, S. D. Odintsov and M. Sasaki, Phys. Rev. D **71**, 123509 (2005)

- [arXiv:hep-th/0504052]; M. Sami, A. Toporensky, P. V. Tretjakov and S. Tsujikawa, Phys. Lett. B **619**, 193 (2005); B. M. Leith and I. P. Neupane, JCAP **0705**, 019 (2007); S. Nojiri, S. D. Odintsov and M. Sami, Phys. Rev. D **74**, 046004 (2006) [arXiv:hep-th/0605039]; T. Koivisto and D. F. Mota, Phys. Lett. B **644**, 104 (2007); M. R. Setare and E. N. Saridakis, Phys. Lett. B **670**, 1 (2008); A. K. Sanyal, arXiv:0710.2450 [astro-ph].
- [13] A. Vikman, Phys. Rev. D **71**, 023515 (2005).
- [14] M. z. Li, B. Feng and X. m. Zhang, JCAP **0512**, 002 (2005) [arXiv:hep-ph/0503268].
- [15] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005) [arXiv:hep-th/0501025].
- [16] S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005) [arXiv:hep-th/0505215].
- [17] L. P. Chimento and R. Lazkoz, Phys. Lett. B **639**, 591 (2006); S. Sur and S. Das, JCAP **0901**, 007 (2009).
- [18] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034]; S. Capozziello, S. Nojiri and S. D. Odintsov, Phys. Lett. B **632**, 597 (2006) [arXiv:hep-th/0507182]; E. Elizalde, S. Nojiri, S. D. Odintsov, D. Saez and V. Faraoni, Phys. Rev. D **77**, 106005 (2008) [arXiv:0803.1311 [hep-th]].
- [19] R. R. Caldwell and M. Doran, Phys. Rev. D **72**, 043527 (2005).
- [20] H. Wei and R. G. Cai, Phys. Lett. B **634**, 9 (2006).
- [21] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B **607**, 35 (2005); Z. K. Guo, Y. S. Piao, X. M. Zhang and Y. Z. Zhang, *ibid.* **608**, 177 (2005); X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006); H. Mohseni Sadjadi and M. Alimohammadi, Phys. Rev. D **74**, 043506 (2006); X. Zhang, *ibid.* **74**, 103505 (2006).
- [22] W. Fang, W. Hu and A. Lewis, Phys. Rev. D **78**, 087303 (2008).
- [23] M. Cataldo and L. P. Chimento, arXiv:0710.4306 [astro-ph].
- [24] B. McInnes, Nucl. Phys. B **718**, 55 (2005); R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005); R. G. Cai, Y. g. Gong and B. Wang, JCAP **0603**, 006 (2006); I. Y. Aref'eva, A. S. Koshelev and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005); G. Kofinas, G. Panotopoulos and T. N. Tomaras, JHEP **0601**, 107 (2006); I. Y. Aref'eva and A. S. Koshelev, *ibid.* **0702**, 041 (2007); L. P. Chimento, R. Lazkoz, R. Maartens and I. Quiros, JCAP **0609**, 004 (2006); P. S. Apostolopoulos and N. Tetradis, Phys. Rev. D **74**, 064021 (2006); Y. f. Cai, M. z. Li, J. X. Lu, Y. S. Piao, T. t. Qiu and X. m. Zhang, Phys.

- Lett. B **651**, 1 (2007); S. F. Wu, A. Chatrabhuti, G. H. Yang and P. M. Zhang, *ibid.* **659**, 45 (2008); K. Nozari and M. Pourghasemi, JCAP **0810**, 044 (2008); K. Nozari, M. R. Setare, T. Azizi and N. Behrouz, arXiv:0810.1427 [hep-th]; J. Sadeghi, M. R. Setare, A. Banijamali and F. Milani, Phys. Lett. B **662**, 92 (2008).
- [25] S. Jhingan, S. Nojiri, S. D. Odintsov, M. Sami, I. Thongkool and S. Zerbini, Phys. Lett. B **663**, 424 (2008) [arXiv:0803.2613 [hep-th]].
- [26] S. Deser and R. P. Woodard, Phys. Rev. Lett. **99**, 111301 (2007).
- [27] P. Singh, Class. Quant. Grav. **22**, 4203 (2005); D. Samart and B. Gumjudpai, Phys. Rev. D **76**, 043514 (2007).
- [28] H. Stefancic, Phys. Rev. D **71**, 084024 (2005); J. Grande, J. Sola and H. Stefancic, JCAP **0608**, 011 (2006).
- [29] M. Kunz and D. Sapone, Phys. Rev. D **74**, 123503 (2006).
- [30] A. A. Andrianov, F. Cannata and A. Y. Kamenshchik, Phys. Rev. D **72**, 043531 (2005).
- [31] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22**, L35 (2005) [arXiv:hep-th/0409177].
- [32] L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008).
- [33] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006) [arXiv:hep-th/0608008]; J. Phys. Conf. Ser. **66**, 012005 (2007) [arXiv:hep-th/0611071]; J. Phys. A **40**, 6725 (2007) [arXiv:hep-th/0610164].
- [34] B. McInnes, JHEP **0208**, 029 (2002).
- [35] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003).
- [36] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
- [37] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007) [arXiv:0707.1941 [hep-th]]; Phys. Rev. D **77**, 026007 (2008) [arXiv:0710.1738 [hep-th]]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, *ibid.* **77**, 046009 (2008); [arXiv:0712.4017 [hep-th]].
- [38] S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007); L. Pogosian and A. Silvestri, Phys. Rev. D **77**, 023503 (2008); S. Nojiri and S. D. Odintsov, Phys. Lett. B **652**, 343 (2007) [arXiv:0706.1378 [hep-th]]; S. Capozziello and S. Tsujikawa, Phys. Rev. D **77**, 107501 (2008).
- [39] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003) [arXiv:hep-th/0307288].
- [40] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).

- [41] S. Nojiri and S. D. Odintsov, *Gen. Rel. Grav.* **38**, 1285 (2006) [arXiv:hep-th/0506212].
- [42] S. Capozziello, S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **632**, 597 (2006) [arXiv:hep-th/0507182].
- [43] V. Sahni, *Prog. Theor. Phys. Suppl.* **172**, 110 (2008).
- [44] S. Nojiri, S. D. Odintsov and M. Sami, *Phys. Rev. D* **74**, 046004 (2006) [arXiv:hep-th/0605039].
- [45] J. D. Barrow, *Class. Quant. Grav.* **21**, L79 (2004); *Phys. Lett. B* **235**, 40 (1990); S. Cotsakis and I. Klaoudatou, *J. Geom. Phys.* **55**, 306 (2005); S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **70**, 103522 (2004) [arXiv:hep-th/0408170]; J. D. Barrow and C. G. Tsagas, *Class. Quant. Grav.* **22**, 1563 (2005); M. P. Dabrowski, *Phys. Rev. D* **71**, 103505 (2005); M. P. Dabrowski, *Phys. Lett. B* **625**, 184 (2005); C. Cattoen and M. Visser, *Class. Quant. Grav.* **22**, 4913 (2005); L. Fernandez-Jambrina and R. Lazkoz, *Phys. Rev. D* **70**, 121503 (2004); **74**, 064030 (2006); P. Tretyakov, A. Toporensky, Y. Shtanov and V. Sahni, *Class. Quant. Grav.* **23**, 3259 (2006); M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, *Phys. Lett. B* **659**, 1 (2008); I. Brevik and O. Gorbunova, *Eur. Phys. J. C* **56**, 425 (2008).
- [46] K. Bamba, S. Nojiri and S. D. Odintsov, *JCAP* **0810**, 045 (2008) [arXiv:0807.2575 [hep-th]]; S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **78**, 046006 (2008) [arXiv:0804.3519 [hep-th]].
- [47] F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **646**, 105 (2007) [arXiv:hep-th/0612220].
- [48] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, United Kingdom, 2003); K. i. Maeda, *Phys. Rev. D* **39**, 3159 (1989).
- [49] S. Capozziello, S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **634**, 93 (2006) [arXiv:hep-th/0512118].
- [50] T. Chiba, *Phys. Lett. B* **575**, 1 (2003).